Accelerated GSE Analytic Geometry B/Advanced Algebra

Unit 7: Rational and Radical Relationships
Unit 7
Rational and Radical Relationships

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* Revised standards indicated in bold red font.
OVERVIEW

In this unit students will:

- Explore Rational and Radical Functions
- Determine rational numbers extend the arithmetic of integers by allowing division by all numbers except zero. Similarly, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial
- Notice the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers
- Investigate the properties of simple rational and radical functions and then expand their knowledge of the graphical behavior and characteristics of more complex rational functions
- Recall and make use of their knowledge of polynomial functions as well as compositions of functions to investigate the characteristics of these more complex rational functions
- Solve equations and inequalities involving rational and radical functions
- Understand that not all solutions generated algebraically are actually solutions to the equations and extraneous solutions will be explored
- Apply these rational and radical functions with an emphasis on interpretation of real world phenomena as it relates to certain characteristics of the rational expressions

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight process standards should be addressed constantly as well. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the tasks listed under “Evidence of Learning” be reviewed early in the planning process. A variety of resources should be utilized to supplement this unit. This unit provides much needed content information, but excellent learning activities as well. The tasks in this unit illustrate the types of learning activities that should be utilized from a variety of sources.
STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics.

KEY STANDARDS

Rewrite rational expressions

MGSE9-12.A.APR.7 Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Create equations that describe numbers or relationships

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear, quadratic, simple rational, and exponential functions (integer inputs only).

MGSE9-12.A.CED.2 Create linear, quadratic, and exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase “in two or more variables” refers to formulas like the compound interest formula, in which \( A = P(1 + r/n)^nt \) has multiple variables.) (Limit to radical and rational functions.)

Understand solving equations as a process of reasoning and explain the reasoning

MGSE9-12.A.REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Interpret functions that arise in applications in terms of the context

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. (Limit to radical and rational functions.)
MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* (Limit to radical and rational functions.)

**Analyze functions using different representations**

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology. (Limit to radical and rational functions.)

MGSE9-12.F.IF.7b Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

MGSE9-12.F.IF.7d Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

**RELATED STANDARDS**

MGSE9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see \(x^4 - y^4\) as \((x^2)^2 - (y^2)^2\), thus recognizing it as a difference of squares that can be factored as \((x^2 - y^2)(x^2 + y^2)\).

MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.*

**Build a function that models a relationship between two quantities**

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

MGSE9-12.F.BF.1a Determine an explicit expression and the recursive process (steps for calculation) from context. *For example, if Jimmy starts out with $15 and earns $2 a day, the explicit expression “\(2x + 15\)” can be described recursively (either in writing or verbally) as “to find out how much money Jimmy will have tomorrow, you add $2 to his total today.”*

\[
J_n = J_{n-1} + 2, J_0 = 15
\]
STANDARDS FOR MATHEMATICAL PRACTICE

Refer to the Comprehensive Course Overview for more detailed information about the Standards for Mathematical Practice.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

ENDURING UNDERSTANDINGS

- Recognize rational functions as the division of two polynomial functions and rewrite a rational expression
- Find the sum, difference, product, and quotient of rational expressions
- Graph rational and radical functions
- Interpret graphs and discover characteristics of rational functions
- Solve rational and radical equations algebraically and graphically
- Solve rational inequalities

ESSENTIAL QUESTIONS

- How can we extend arithmetic properties and processes to algebraic expressions and how can we use these properties and processes to solve problems?
- How do the polynomial pieces of a rational function affect the characteristics of the function itself?
- How are horizontal asymptotes, slant asymptotes, and vertical asymptotes alike and different?
• Why are all solutions not necessarily the solution to an equation? How can you identify these extra solutions?
• Why is it important to set a rational inequality to 0 before solving?

CONCEPTS/SKILLS TO MAINTAIN

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- Computation with fractions
- Factoring polynomials
- Solving linear and quadratic equations

SELECTED TERMS AND SYMBOLS

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

The definitions below are for teacher reference only and are not to be memorized by the students. Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

The websites below are interactive and include a math glossary suitable for elementary children. Note – At the elementary level, different sources use different definitions. Please preview any website for alignment to the definitions given in the frameworks.

This web site has activities to help students more fully understand and retain new vocabulary.

http://intermath.coe.uga.edu/dictnary/homepg.asp
Definitions and activities for these and other terms can be found on the Intermath website.

- **Algebra:** The branch of mathematics that deals with relationships between numbers, utilizing letters and other symbols to represent specific sets of numbers, or to describe a pattern of relationships between numbers.

- **Coefficient:** A number multiplied by a variable.

- **Equation:** A number sentence that contains an equality symbol.
• **Expression**: A mathematical phrase involving at least one variable and sometimes numbers and operation symbols.

• **Extraneous Solutions**: A solution of the simplified form of the equation that does not satisfy the original equation.

• **Inequality**: Any mathematical sentence that contains the symbols > (greater than), < (less than), ≤ (less than or equal to), or ≥ (greater than or equal to).

• **Polynomial**: A mathematical expression involving the sum of terms made up of variables to nonnegative integer powers and real-valued coefficients.

• **Radical Function**: A function containing a root. The most common radical functions are the square root and cube root functions, \( f(x) = \sqrt{x} \) and \( g(x) = \sqrt[3]{x} \).

• **Rational Function**: The quotient of two polynomials, \( P(z) \) and \( Q(z) \), where \( R(z) = \frac{P(z)}{Q(z)} \).

• **Reciprocal**: Two numbers whose product is one. For example, \( m \times \frac{1}{m} = 1 \).

• **Variable**: A letter or symbol used to represent a number.

**EVIDENCE OF LEARNING**

By the conclusion of this unit, students should be able to demonstrate the following competencies:

- Rewrite rational expressions in different forms
- Add, subtract, multiply, and divide rational expressions
- Solve rational and radical equations
- Solve rational inequalities
- Graph rational and radical functions and identify key characteristics
- Interpret solutions to graphs and equations given the context of the problem
Operations with Rational Expressions

Thinking about operations with rational numbers, or fractions, will help us perform addition, subtraction, multiplication, and division with rational expressions. We will use examples involving fractions to help us extend our thinking to dealing with fractions with variables, or rational expressions.

- Simplifying Rational Expressions

Think about the fraction $\frac{108}{210}$. What operation do we use to rewrite this fraction in simplest form?

What is the possible obstacle in using this operation to simplify fractions?

Let’s try to simplify another way. Find the prime factorization of the numerator and denominator of the fraction above. Use this form of the numerator and denominator to quickly simplify the fraction.

Now let’s think about the fraction: $\frac{x^2-9}{x^2+7x+12}$. How can we use the idea of prime factorization to help us simplify this rational expression?

Try this one: $\frac{4-x}{x-4}$ How can factoring help us simplify this rational expression?

- Multiplying and Dividing Rational Expressions

We now need to think about multiplying fractions. Take a minute to discuss with a partner how you would solve the following problem. Try to find more than one way and show your results below:

\[
\begin{array}{c}
\frac{4}{14} \cdot \frac{24}{10}
\end{array}
\]
Which method from above do you think would be easiest to extend to multiplication of rational expressions? Why?

Take a look at \( \frac{5x^2}{x^2-4} \cdot \frac{x+2}{10x^3} \). Use your ideas from above to help you multiply these two fractions.

Now try this one: \( \frac{3x+6}{x^2-9} \cdot \frac{4x+12}{6x^2+12x} \)

What if we change the fraction multiplication problem that we started with to a division problem? Talk to your partner about how to solve the problem below:

\[ \frac{4}{14} ÷ \frac{24}{10} \]

What is the one difference in solving a fraction division problem versus a fraction multiplication problem?

Apply that idea to this problem: \( \frac{4x+8}{8x} ÷ \frac{x^2-4}{6x^2} \)

Let’s try one more: \( \frac{x^2-2x-15}{3x^2+12x} ÷ \frac{x^2-9}{x^2+4x} \)

- Adding and Subtracting Rational Expressions

The idea of using the processes for operations with fractions to guide us as we operated with rational functions continues, but addition and subtraction may seem a little more involved. Just like with fractions it is necessary to have common denominators in both rational expressions before you can add or subtract. Think about the fraction addition problem \( \frac{3}{10} + \frac{1}{6} \). What is the least common denominator (LCD)?

You might be able to quickly realize that 30 is the LCD, but why is it 30? Turn to your partner and explain a couple of ways of finding a common denominator.
When thinking about denominators like $x + 2$ or $x - 3$ it becomes important to understand what makes a LCD. In the fraction problem above, you might have been able to say that 30 is the LCD because it is the smallest number that both 10 and 6 divide into, but how do you create that number if it isn’t obvious? (Hint: Think about prime factorization.)

When dealing with rational expressions, factoring is key. You must find all of the factors of each denominator to know what the LCD should be. Let’s try some. Find the LCD for the following problems:

a. $\frac{3}{5a} \cdot \frac{b}{4a^2}$

b. $\frac{4}{x+5} \cdot \frac{3}{x-5}$

c. $\frac{2x}{x+2} \cdot \frac{x+1}{x^2-3x-10}$

d. $\frac{7}{x} \cdot \frac{5}{2x^2+3x}$

e. $\frac{x-9}{x^2+8x+16} \cdot \frac{x}{x^2+7x+12}$

Once you find the LCD, you complete the operation just like you would with fractions. Try these problems:

f. $\frac{3}{x+3} + \frac{2}{x-3}$

g. $\frac{6x+7}{x^2-4} + \frac{2}{x-2}$

h. $\frac{10}{6m^2} - \frac{2n}{5m^3}$

i. $\frac{6}{8a+4} + \frac{3a}{8}$

j. $\frac{x}{x^2+5x+6} - \frac{2}{x^2+3x+2}$
Characteristics of Rational Functions

Now that we have worked with rational expressions, it is time to look at rational functions themselves. Since a rational function is the quotient of two polynomial functions it is important to first look at the characteristics of the individual polynomials.

Let’s investigate \( g(x) = x^2 + 3x - 10 \). What facts can you write about \( g(x) \)?

What is the Domain? How do you determine the Domain?

What is the Range? How do you determine the Range?

Where are the Roots or Zeros found? What are some different ways you know to find them?

What is the End Behavior? How do you know?

Let’s investigate \( f(x) = x + 1 \). What facts can you write about \( f(x) \)?

What is the Domain?

What is the Range?

What are the Roots or Zeros?

What is the End Behavior? How do you know?

Now let’s consider the case of the rational function \( r(x) = \frac{f(x)}{g(x)} \) where \( f \) and \( g \) are the polynomial functions above. Write the expression for the function \( f(x) \).

What is the domain of \( r(x) \)? Which function, \( f \) or \( g \), affects the domain the most? Why?

What do you think the range of \( r(x) \) will be? Why is this so difficult to determine?

What are the roots or zeros of \( r(x) \)? Which function helps you find them?

What do you think the end behavior will be? Why?

Where will \( r(x) \) intersect the \( y \)-axis? How do you know?

Now let’s look at the graph of \( r(x) \) using your calculator.
Georgia Department of Education
Georgia Standards of Excellence Framework
Accelerated GSE Analytic Geometry B/Advanced Algebra • Unit 7

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Richard Woods, State School Superintendent
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What value does \( r(x) \) approach as \( x \) approaches infinity? Negative infinity? How could you describe this behavior?

What occurs at the \( x \)-values when \( g(x) = 0 \)? Do you think this will happen every time the denominator is equal to zero?

At what \( x \)-values does \( r(x) \) change signs (either + to – or vice-versa)? What else occurs at these \( x \)-values?

Based on the graph from your calculator, what is the range of \( r(x) \)?

When is the function increasing? decreasing? Are there any local extrema? How can you be certain?

Let’s try a few more problems and see if we can discover any patterns…

1. Let \( f(x) = 5 \) and \( g(x) = x^2 - 6x + 8 \). Let \( r(x) = \frac{f(x)}{g(x)} \).

What is the domain of \( r(x) \)? Which function, \( f \) or \( g \), affects the domain the most? Why?

What do you think the range of \( r(x) \) will be?

What are the roots or zeros of \( r(x) \)? Which function helps you find them?

What do you think the end behavior will be? Why?

Where will \( r(x) \) intersect the y-axis? How do you know?

Now let’s look at the graph of \( r(x) \) using your calculator.

What value does \( r(x) \) approach as \( x \) approaches infinity? Negative infinity? How could you describe this behavior? Why do you think this happens?

What occurs at the \( x \)-values when \( g(x) = 0 \)? Do you think this will happen every time the denominator is equal to zero?

At what \( x \)-values does \( r(x) \) change signs (either + to – or vice-versa)? What else occurs at these \( x \)-values?

When is the function increasing? decreasing? Are there any local extrema? How can you be certain?
Based on the graph from your calculator, what is the range of \( r(x) \)?

2. Let \( r(x) = \frac{2x^2 + 7x - 4}{x^3 - 1} \).

What is the domain of \( r(x) \)?

What do you think the range of \( r(x) \) will be?

What are the roots or zeros of \( r(x) \)?

What do you think the end behavior will be?

Where will \( r(x) \) intersect the y-axis? How do you know?

Now let’s look at the graph of \( r(x) \) using your calculator.

What value does \( r(x) \) approach as \( x \) approaches infinity? Negative infinity? How could you describe this behavior? Why do you think this happens?

What occurs at the \( x \)-values when the denominator is equal to zero? Do you think this will happen every time?

At what \( x \)-values does \( r(x) \) change signs (either + to – or vice-versa)? What else occurs at these \( x \)-values?

When is the function increasing? decreasing? Are there any local extrema? How can you be certain?

Based on the graph from your calculator, what is the range of \( r(x) \)?

3. Let \( r(x) = \frac{4x + 1}{4 - x} \).

What is the domain of \( r(x) \)?

What do you think the range of \( r(x) \) will be?

What are the roots or zeros of \( r(x) \)?
What do you think the end behavior will be?

Where will \( r(x) \) intersect the \( y \)-axis? How do you know?

Now let’s look at the graph of \( r(x) \) using your calculator.

What value does \( r(x) \) approach as \( x \) approaches infinity? Negative infinity? How could you describe this behavior? Why do you think this happens?

What occurs at the \( x \)-values when the denominator is equal to zero? Do you think this will happen every time?

At what \( x \)-values does \( r(x) \) change signs (either + to – or vice-versa)? What else occurs at these \( x \)-values?

When is the function increasing? decreasing? Are there any local extrema? How can you be certain?

Based on the graph from your calculator, what is the range of \( r(x) \)?

**Now let’s summarize our findings and conclusions.**

When is the domain of a rational function not \((-\infty, \infty)\)? So what is your advice on how to determine the domain of a rational function?

Is the range of a rational function difficult to find? Why or why not?

How do you find the zeros or roots of a rational function?

How do you know where to find vertical asymptotes?

What does a horizontal asymptote tell you about a rational function? Are they easy to locate? Do you know of any shortcuts to find them?

How do you know where a rational function will intersect the \( y \)-axis? Will a rational function always have a \( y \)-intercept? Can you give an example?
What possible things could occur at the x-values where a rational function changes signs?

Do you think it would be possible to use all of our knowledge of rational functions to create sketch without using the graphing calculator? Can you explain how this would work to another classmate?
Horizontal Asymptotes: How do we find them?

As we discuss the characteristics of rational functions, we know that it is important to consider the properties of the individual functions. Knowing about the individual functions helps us to know about the rational function. But as we discuss the details, let us consider the range values of the rational function. To do this it will be important consider the range values through a table and the graph. But to do this we are going to look at very large values for x.

1. Let \( f(x) = 5 \) and \( g(x) = x^2 - 6x + 8 \). Let \( r(x) = \frac{f(x)}{g(x)} \). Complete the table of values of \( r(x) \).

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What type of trends do you see from the y-values of this function?

As the x-values head toward infinity, is there any significance to the y-values?

Examine the graph in your graphing utility to get a better picture.

How does the graph relate to your table?
2. Let \( r(x) = \frac{2x^2 + 7x - 4}{x^3 - 1} \). Complete the table of values of \( r(x) \).

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What type of trends do you see from the \( y \)-values of this function?

As the \( x \)-values head toward infinity, is there any significance to the \( y \)-values?

Examine the graph in your graphing utility to get a better picture.
How does the graph relate to your table?

3. Let \( r(x) = \frac{4x+1}{4-x} \). Complete the table of values of \( r(x) \).

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What type of trends do you see from the \( y \)-values of this function?
As the x-values head toward infinity, is there any significance to the y-values?

Examine the graph in your graphing utility to get a better picture.

How does the graph relate to your table?

4. Let \( r(x) = \frac{3x^2 + 27}{2x^2 - 6x - 8} \).

Complete the table of values of \( r(x) \).

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What type of trends do you see from the y-values of this function?

As the x-values head toward infinity, is there any significance to the y-values?

Examine the graph in your graphing utility to get a better picture.

How does the graph relate to your table?
5. Consider the range of the function, \( R(x) = \frac{x^2 - 2x + 5}{x - 6} \). Complete the table of values of \( r(x) \).

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<td>-200</td>
<td></td>
</tr>
<tr>
<td>-300</td>
<td></td>
</tr>
<tr>
<td>-700</td>
<td></td>
</tr>
<tr>
<td>-1000</td>
<td></td>
</tr>
</tbody>
</table>

What type of trends do you see from the \( y \)-values of this function?

As the \( x \)-values head toward infinity, is there any significance to the \( y \)-values?

Examine the graph in your graphing utility to get a better picture. How does the graph relate to your table?

Looking at the function how is this one different from the others that we have considered?

Let’s look back at the first task of this unit to help us find the slant asymptote. We can use long division to help us find the equation.

**Now let’s summarize our findings and conclusions.**

When is the domain of a rational function not \((-\infty, \infty)\)? So what is your advice on how to determine the domain of a rational function?

Is the range of a rational function difficult to find? Why or why not?
How do you find the zeros or roots of a rational function?

How do you know where to find vertical asymptotes?

What does a horizontal asymptote tell you about a rational function? Are they easy to locate? Do you know of any shortcuts to find them?

How do you know where a rational function will intersect the y-axis? Will a rational function always have a y-intercept? Can you give an example?

What possible things could occur at the x-values where a rational function changes signs?

Do you think it would be possible to use all of our knowledge of rational functions to create an accurate sketch without using the graphing calculator? Can you explain how this would work to another classmate?
Graphing Rational Functions without a calculator

Let \( r(x) = \frac{3x^2 + 27}{2x^2 - 6x - 8} \).

What is the domain of \( r(x) \)?

What are the roots or zeros of \( r(x) \)?

What do you think the end behavior will be?

Where will \( r(x) \) intersect the \( y \)-axis? How do you know?

Are there any vertical asymptotes? If so, where are they located?

Is there a horizontal asymptote? If so, where is it located?

At what \( x \)-values should \( r(x) \) change signs (either + to – or vice-versa)? Why?

Where is \( r(x) > 0 \)? Where is \( r(x) < 0 \)? (Hint: use a sign chart)

Now let’s try to sketch the graph of \( r(x) \) without using your calculator.

Based on your sketch, what do you think the range of \( r(x) \) will be?
Now let’s compare your sketch to the graph of \( r(x) \) using your calculator. How did your sketch match up with the actual curve? What characteristics of the graph were you not able to anticipate?

When is the function increasing? decreasing? Are there any local extremum? How can you be certain?

Based on the graph from your calculator, what is the range of \( r(x) \)?

Try this one next. Let \( r(x) = \frac{x^3 + 1}{3x^3 - 27x} \)

What is the domain of \( r(x) \)?

What are the roots or zeros of \( r(x) \)?

What do you think the end behavior will be?

Where will \( r(x) \) intersect the y-axis? How do you know?

Are there any vertical asymptotes? If so, where are they located?

Is there a horizontal asymptote? If so, where is it located?

At what \( x \)-values should \( r(x) \) change signs (either + to – or vice-versa)? Why?

Where is \( r(x) > 0 \)? Where is \( r(x) < 0 \)?

Now let’s try to sketch the graph of \( r(x) \) without using your calculator.

Based on your sketch, what do you think the range of \( r(x) \) will be?
Now let’s compare your sketch to the graph of \( r(x) \) using your calculator. How did your sketch match up with the actual curve? What characteristics of the graph were you not able to anticipate?

When is the function increasing? decreasing? Are there any local extremum? How can you be certain?

Based on the graph from your calculator, what is the range of \( r(x) \)?

In your own words, describe the process that you would go through in order to create a sketch of any rational function.
Jogging into the Wind

Lisa is quite an athlete, but sometimes trying to get to work in the windy city of Chicago can be a big challenge. Lisa always jogs from her condominium to her office in downtown Chicago, and this distance is 1.75 miles. Lisa likes to keep a steady pace of 704 feet per minute. Unfortunately, Lisa lives directly west of her office, which means her morning jog to work always puts her directly into the wind coming off of Lake Michigan.

(a) Let $s$ be the speed of the wind in feet per minute. Write an expression for $r(s)$, the speed at which Lisa is moving relative to the total distance of her journey, in terms of $s$.

(b) Lisa wants to know how long it will take her to jog to work. Write an expression for $T(s)$, the time it will take in minutes, in terms of $s$.

(c) What is the vertical intercept of $T$? What does this point represent in terms of Lisa’s jog to work?

(d) At what value of $s$ does the graph have a vertical asymptote? Explain why this makes sense in this situation.

(e) For what value of $s$ does $T(s)$ make sense in the context of this problem?
Lisa has been training for a marathon, and now she maintains a constant speed of 720 feet per minute when jogging to work.

(f) On a particular day, Lisa guesses that the wind is blowing at 4.25 miles per hour against her. How long will it take Lisa to get to work?

(g) Obviously, Lisa doesn’t really know the speed of the wind. Make a table showing the time it will take her to get to work against the various wind resistances:

<table>
<thead>
<tr>
<th>Speed of wind (Feet per minute)</th>
<th>Lisa’s speed (Feet per minute)</th>
<th>Time for Lisa to travel 1.75 miles to work (Minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>720</td>
<td></td>
</tr>
<tr>
<td>176</td>
<td></td>
<td></td>
</tr>
<tr>
<td>352</td>
<td></td>
<td></td>
</tr>
<tr>
<td>528</td>
<td></td>
<td></td>
</tr>
<tr>
<td>704</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(h) Sketch a graph of the equation from part (g). Explain why $s = 720$ does not make sense for this function, both in terms of the jogging trip and in terms of the equation.
Hank’s Hot Dog Stand

Hank runs a successful hot dog stand right across from the arch at the University of Georgia in downtown Athens. Hank has to order his hot dogs, buns, mustard, relish, and all other condiments in bulk, as well as pay taxes, licensing fees, and other small business expenses. Therefore, Hank has a relatively large “sunk” cost associated with his business. The cost of producing \( h \) hot dogs is given by

\[
C(h) = 2750 + 0.45h
\]

(a) Hank wants to figure out how much to charge a customer for a hot dog if he wishes to make a $0.25 profit on each hot dog sold. Suppose Hank sold 100 hot dogs in an afternoon. What is the cost of making this many hot dogs? How much is this per hot dog? What should Hank charge per hot dog?

(b) Hank wants to analyze what his cost per hot dog would be for different levels of sales. Complete the table below showing his costs at these different levels.

<table>
<thead>
<tr>
<th>Number of Hot Dogs Sold</th>
<th>0</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>10,000</th>
<th>100,000</th>
<th>1,000,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Cost</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cost per Hot Dog</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hank Should Charge?</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Explain why the average cost per hot dog levels off.

(d) Find an equation for the average cost per hot dog of producing \( h \) hot dogs.

(e) Find the domain of the average cost function.
(f) Using the data points from your table above, sketch the average cost function. How does the graph reflect that the average cost levels off?

![Graph of average cost function]
That’s Radical Dude:

Let’s explore radical functions. By definition, a radical function is one that contains any sort of radical. We are going to explore two of the more common radical functions, the square root and the cube root.

Complete the table of value for the function, $f(x) = \sqrt{x}$. This is the square root function.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What did you notice about some of the values? If you typed the function into a calculator and tried to evaluate it for some of the $x$-values, what message appeared? Why?

Graph the function in the grid provided below.

What is the domain of this function?

What is the range of the function?
Complete the table of values for the function, \( f(x) = \sqrt{x + 2} \)

<table>
<thead>
<tr>
<th>x</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What did you notice about some of the values? If you typed the function into a calculator and tried to evaluate it for some of the x-values, what message appeared? Why?

Graph the function in the grid provided below:

What is the domain of this function?

What is the range of the function?
Complete the table of values for the function, \( f(x) = \sqrt{9 - x^2} \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What did you notice about some of the values? If you typed the function into a calculator and tried to evaluate it for some of the \( x \)-values, what message appeared? Why?

Graph the function in the grid provided below.

What is the domain of this function?

What is the range of the function?

Using the three examples above, make a conjecture about the domain of a radical function.

Use your conjecture to determine the domain of this function, \( f(x) = \sqrt{2x + 5} \), without graphing it. Check your solution by graphing it on a graphing calculator.

Now let’s look at another common radical function, the cube root.
Complete the table of values for the function, \( f(x) = \sqrt[3]{x} \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-8</th>
<th>-6</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Do you get any of the same error messages for this function that you did in the table of values for the square root function? Why do you think that is so?

Graph the function in the grid provided below.

What is the domain of this function?
What is the range of the function?
Let’s Get to “Work”:

Rational Equations can be used to model some interesting real life phenomena. Distance, rate, and time problems as well as multi-person work problems are particularly suited to be modeled with a rational equation.

“Work” Problems

Two are better than one when it comes to completing a job. We can use rational equations to help us figure just how much better two can do a job than one person acting alone.

Jonah can paint a house by himself in 12 hours. Steve can do the same job in eight hours. How long will it take them to complete the job together? To help us answer this question we first need to think about how much of the job Jonah and Steve can do in one hour.

Let’s let $t$ = hours of work it takes to do job together. So to find out how much of the job they complete together in one hour, we will use this equation:

$$\frac{1}{12} + \frac{1}{8} = \frac{1}{t}$$

Now we have to think about how to solve this equation. There are actually a few different ways to approach it. Talk to your partner to see what ways you can come up with together.

Equations are balanced statements and can be easily changed as long as you make sure to perform the same operation to every piece of the equation. This idea lets us multiply the equation by the LCD in order to create a simpler equation to solve. Try it with the equation above to see how long it will take Jonah and Steve to paint the house together.

Let’s try another problem. Paul can paint a room two times as fast as Jamie. Working together they can paint the room in three hours. How long would it take each of them to paint the room alone?
Average Cost

Have you ever wondered why large retailers such as Wal-Mart can offer products as such low costs? The secret is in the quantity that they purchase. To understand this idea we are going to explore average cost.

At the Stir Mix-A-Lot blender company, the weekly cost to run the factory is $1400 and the cost of producing each blender is an additional $4 per blender.

a. Write a function rule representing the weekly cost in dollars, $C(x)$, of producing $x$ blenders.
b. What is the total cost of producing 100 blenders in one week?
c. If you produce 100 blenders in one week, what is the total production cost per blender?
d. Will the total production cost per blender always be the same? Justify your answer.
e. Write a function rule representing the total production cost per blender $P(x)$ for producing $x$ blenders.
f. Using your graphing calculator, create a graph of your function rule from part e. Does the entire graph make sense for this situation? If not, what part does?
g. What is the production cost per blender if 300 blenders are produced in one week? If 500 blenders are produced in one week?
h. What happens to the total production cost per blenders as the number of blenders produced increases? Explain your answer.
i. How many blenders must be produced to have a total production cost per blender of $8$?
j. The function for the production cost of the blenders is a rational function. What other information can we gather about this situation based on the characteristics of rational functions?

Pendulum

Tommy visited the Museum of History and Technology with his class. They saw Foucault’s Pendulum in Pendulum Hall and it was fascinating to Tommy. He knew from science class that the time it takes a pendulum to complete a full cycle or swing depends upon the length of the pendulum. The formula is given by $T = 2\pi \sqrt{\frac{L}{32}}$ where $T$ represents the time in seconds and $L$ represents the length of the pendulum in feet. He timed the swing of the pendulum with his watch and found that it took about 8 seconds for the pendulum to complete a full cycle. Help him figure out the length of the pendulum in feet.
Tommy thought that a pendulum that took a full 20 seconds to complete a full cycle would be very dramatic for a museum. How long must that pendulum be? If ceilings in the museum are about 20 feet high, would this pendulum be possible?
Extraneous Solutions:

Rational Equations are useful to help you solve some real-life problems like work and average cost. Because those problems have real-life context, issues such as zeros in the denominator can be avoided because it doesn’t make sense to solve for 0 hours of work or for 0 blenders. When you look at a rational equation algebraically, though, you have to watch out for such issues. Solutions that create a zero in the denominator are known as extraneous solutions. Let’s work an example.

Solve \( \frac{2m}{m-1} + \frac{m-5}{m^2-1} = 1 \)

You should get two solutions. Do both solutions give you a balanced equation? Check each one by substituting the value back into the original equation. Do you see any problems?

It is important to always check rational equations for extraneous solutions or restrictions on your domain. Not every solution that you find works!

Let’s practice a few more:

a. \( \frac{3}{5x} + \frac{7}{2x} = 1 \)

b. \( \frac{4}{k^2-8k+12} = \frac{k}{k-2} + \frac{1}{k-6} \)

c. \( \frac{2b-5}{b-2} - 2 = \frac{3}{b+2} \)

d. \( \frac{1}{x-2} = \frac{x}{2x-4} + 1 \)

e. \( \frac{1}{x} + \frac{1}{x-3} = \frac{x-2}{x-3} \)

Rational equations aren’t the only type of equations that can create extraneous solutions. Radical equations can also produce extraneous solutions. Let’s look at a simple example first. Take the equation \( \sqrt{x} = -3 \). One way to find solutions is to graph both sides of the equation to find the point of intersection. Do that now. What do you notice?

Now let’s solve this equation algebraically. The key to solving a radical equation is to isolate the radical on one side and then square both sides of the equation in order to eliminate the radical. Do that now for the equation above. What appears to be the answer? Is it really the answer? How can we check other than looking at the graph?
Some radical equations do have solutions. Try solving \( x = 1 + \sqrt{5x - 9} \). Make sure to check your answers for any extraneous solutions.

Let’s practice a few more:

\[
\begin{align*}
\text{f.} & \quad \sqrt{2x - 1} + 5 = 2 \\
\text{g.} & \quad x - 3 = \sqrt{30 - 2x} \\
\text{h.} & \quad \sqrt{5x + 3} = \sqrt{3x + 7} \\
\text{i.} & \quad 2\sqrt{x + 8} = 3\sqrt{x - 2}
\end{align*}
\]
To Bracket or Not To Bracket…That is the Question

Rational Inequalities

Solving rational inequalities is very similar to solving polynomial inequalities. This means that we are trying to figure out when the function itself has a value greater than or less than 0. But because rational expressions have denominators (and therefore may have places where they're not defined), you have to be a little more careful in finding your solutions.

To solve a rational inequality, you first find the zeroes (from the numerator) and the undefined points (from the denominator). You use these zeroes and vertical asymptotes to divide the number line into intervals. Then you find the sign of the rational function on each interval. The final solution of the inequality should be written using interval notation.

Let’s try one: \( \frac{x-1}{x+5} \geq 0 \)

Now that we’ve worked through an entire problem, think about this type of problem: \( \frac{5x+1}{x} < 1 \). How is this problem different from our first problem? What should we do so that we can solve it?

Now complete the problem.

John is struggling with the same problem that you just completed. His idea was to multiply both sides by \( x \) so he could “clear the fraction”. His answer doesn’t match the one we found above. Explain why his method doesn’t work.

Jasmine is struggling with interval notation. She remembers from polynomials that the symbols \( \geq \) and \( \leq \) require different notation than the symbols \( > \) and \( < \) do, but she isn’t sure how that will affect rational inequalities. She is working on the problem \( \frac{x+3}{x-5} \leq 0 \) and she thinks the final answer should be \([-3, 5]\). Kendra comes by and looks at Jasmine’s paper. Kendra thinks the answer should be \([-3, 5)\). Who is right and why?
Culminating Task: NFL Passer Rating: Application of Rational Functions

The National Football League has developed a rating system for quarterbacks based upon a number of different factors including passing attempts, completions, yards earned, touchdowns, and interceptions. These factors can be combined into one rational function which will give the Quarterback Passer Rating or QB Rating based upon passing attempts in a game. The QB Rating function is

$$ R = \frac{25A + 1000C + 50Y + 4000T - 5000I}{12A} $$

where $R$ is the QB Rating, $A$ are attempts per game, $C$ are completions per game, $Y$ are the total passing yards per game, $T$ are the touchdowns per game, and $I$ are the interceptions per game. These statistics can also be looked at using yearly totals to calculate a season long QB Rating.

In the 2008 NFL season, Peyton Manning, quarterback of the Indianapolis Colts, completed 371 out of 555 passes he threw during a 16 game season. He threw for 4002 yards, 27 touchdowns, and 12 interceptions. Using his complete season statistics, calculate Peyton Manning’s QB Rating for the 2008 season.

Now, find the average number of completions, attempts, yards, touchdowns, and interceptions per game in the 2008 season. Calculate his QB Rating using these average values for the season and compare to the previous rating. Analyze your findings as to any similarities or differences.

Looking at the solutions for each calculation of $R$, determine if the solutions were similar or different. Explain why the ratings were the same or different.

Peyton is looking to improve his QB Rating for next season, and he feels very confident in his ability to repeat most of his statistics. He knows that if he can improve his completion percentage (passes completed / passes attempted) then he should be able to improve his QB Rating.

Suppose we let all of Peyton’s statistics for 2009 be the same as in 2008 except for the number of attempts, which we will leave as $A$.

Write the formula for $R$ in terms of $A$.

How many attempts will Peyton need in 2009 to generate a Rating > 100? Is this more or less attempts than in 2008? Is this feasible or actually possible? Why or why not?

How many attempts will generate a Rating < 75? Is this more or less attempts than in 2008?
What kind of a relationship is there between Peyton’s QB Rating and the number of attempts that he makes during the season?

How many attempts will generate a Rating < 50? What do you think the absolute minimum QB Rating could be for Peyton in 2009? Is this feasible? Why or why not?

Based on our formula, is there a maximum QB Rating for Peyton in 2009? How did you determine this number?

Find the inverse function for $R(A)$. What information would the inverse function give us that the original does not? Would this be helpful in answering the questions above?

What would be the completion percentage needed in order for Peyton to have a QB rating of 125 in 2008 (assuming all other statistics remain constant)?
Culminating Task: Create A Can

The 10 O’clock coffee company is creating a can to market their coffee. They are concerned with Going Green in their production as well as being cost efficient. They are using recycled aluminum and need it to hold 135 cubic inches. The top and bottom of the can is made with a thicker aluminum that cost 6 cents per square inch, while the sides are manufactured with a thinner grade that costs 4 cents per square inch.

(This task was modified from the Pearson – Addison Wesley textbook seventh edition.)

Part I:

A. Determine a formula for the volume of the can in terms of h, r, and π.

B. Create a net diagram for the coffee can. Determine a formula for the cost of the can based upon your net diagram.

C. Find an equation for the cost of the coffee can that includes all the requirements mentioned in the problem scenario.

D. What is the minimum cost of the coffee can? What are the dimensions of this coffee can?

Part II:

Pick a cylindrical can product of your own and determine its radius and height. Let’s assume the same 3 to 2 cost ratio for the lids and side of your can as in Part I, but let’s make the material much cheaper (.6 cents for a lid and .4 cents for the side). Investigate how the dimensions of your product fit the minimum cost model you created in Part I. Obviously, the volume of your can may be different so your model will need to be adjusted to fit the new information. In addition, explain why there may be differences between your results and the actual product.